Energy-Gap Function in the Theory of Superconductivity

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The relation between the form of the energy-gap function $\Delta(\epsilon)$ and experimental results for the superconductors with stronger coupling, Pb and Hg, is discussed. It is shown that the critical field curve is affected rather strongly by the form of $\Delta(\epsilon)$ near the Fermi surface, but that the form of $\Delta(\epsilon)$ at distances of a Debye energy or more has little effect. The positive deviation of the critical field from a parabolic indicates that Δ is flat or slightly increasing on moving away from the Fermi surface. Solutions of the BCS integral equation are given for the Bogoliubov, Bardeen-Pines, and Eliashberg interactions. When a screened Coulomb interaction is included, all of these solutions have the necessary form at the Debye energy to explain the anomalous tunneling behavior of Pb. The critical field data eliminate both the Bogoliubov and Bardeen-Pines interactions, and favors the Eliashberg interaction. However, the Pb and Hg critical fields can not be reproduced with these solutions since the ratio $\Delta(0)/kT_c$ is much too small.

INTRODUCTION

UNTIL recently, much of the experimental work in the field of superconductivity has either been directly concerned with the measurement of the gap in the single-particle excitation spectrum of the superconductor, or at least the results are interpretable in terms of such an energy gap.[1] In the BCS theory,[2] as well as in the version of Bogoliubov,[3] the energy gap is the value of a function $\Delta(\mathbf{k})$ at the Fermi surface $|\mathbf{k}|=k_F$. [Actually the gap is $2\Delta(k_F)$.] In the original treatments[2,3] of the theory, the effective electron-electron interaction was treated as a constant and this led to a constant value of $\Delta(\mathbf{k})$—denoted by ϵ_0 in BCS.

Numerical calculations,[4,5] as well as an approximate analytic solution,[6] using more complicated (and presumably more accurate) interactions have led to energy-gap functions which depend on the energy, $\epsilon(k)$, in a rather complicated way.[7] However, in the weak coupling limit the predictions for measurable quantities are identical to those obtained by BCS with their simple interaction, showing that the theory does give a law of corresponding states. Part of the reason that the results are independent of the interaction is due to the fact that $\Delta(\epsilon)$ has a vanishing first derivative at the Fermi surface. Furthermore, the distance one must go away from the Fermi surface in order to have an order of magnitude change in Δ is of the order of $k\theta_D$, the Debye energy. On the other hand, the calculations of the physically measurable quantities involve integrals over regions of the order of kT_c or Δ_0 (the value of Δ at the Fermi surface and at $T=0°$) in width about the Fermi surface. In the weak coupling limit for which $\Delta_0 \ll k\theta_D$, the energy-gap function does not vary much in the interesting region and can thus be treated as constant.

On the other hand, if the coupling is not weak so that the energy gap is not very small compared to $k\theta_D$, it is possible that the energy dependence of Δ may manifest itself in the physical behavior of the superconductor. If this were the case, then the results would depend on the form of the interaction, and one would expect deviations from the law of correspondent states. Two superconductors with relatively large ratios of $\Delta_0/k\theta_D$ are Pb and Hg; these are also the superconductors with the largest experimental deviations from the law of corresponding states.

Recently Giaever, Hart, and Megerle[8] have obtained anomalous results for the tunneling from superconducting lead through an insulator into a normal metal. They interpret the anomaly in terms of the effect of the variation of $\Delta(\epsilon)$ in the Pb on the density of states in the region of the Debye energy. This seems to be the first fairly direct experimental information about the nature of the energy-gap function away from the Fermi surface.

Another interesting result is that of Finnemore and Mapother[9] in which they found they could fit their experimental critical field curves for Hg by assuming a constant energy-gap function and treating $2\Delta_0/kT_c$ as an adjustable parameter. The energy gap they obtain is in reasonable agreement with that from infrared absorption. Their calculated results are in contrast to those of the author[4] in which a solution was found using a Bardeen-Pines type of interaction[10] of strong enough

[1] For a review and references, see J. Bardeen and J. R. Schrieffer, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1961) Vol. 3, p. 170. The most recent method of determining the gap is in the tunneling experiments of I. Giaever and K. Megerle, Phys. Rev. **122**, 1101 (1961) and of S. Shapiro, P. H. Smith, J. Nicol, J. L. Miles, and P. F. Strong, IBM J. Res. Develop. **6**, 34 (1962). See also reference 8.

[2] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957); referred to as BCS.

[3] N. N. Bogoliubov, Nuovo Cimento **7**, 794 (1958); N. N. Bogoliubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau Enterprises Inc., New York, 1959).

[4] J. C. Swihart, IBM J. Research Develop. **6**, 14 (1962).

[5] G. J. Culler, B. D. Fried, R. W. Huff, and J. R. Schrieffer, Phys. Rev. Letters **8**, 399 (1962).

[6] P. Morel and P. W. Anderson, Phys. Rev. **125**, 1263 (1962).

[7] In references 4–6 as well as in this paper, only the spherically symmetric part of Δ is considered; that is, Δ is taken as a function of $|\mathbf{k}|$ or of the normal-state single-electron energy, ϵ.

[8] I. Giaever, H. R. Hart, Jr., and K. Megerle, Phys. Rev. **126**, 941 (1962). See also reference 18.

[9] D. K. Finnemore and D. E. Mapother, Bull. Am. Phys. Soc. **7**, 175 (1962); referred to as FM.

[10] J. Bardeen and D. Pines, Phys. Rev. **99**, 1140 (1955); H. Frohlich, Proc. Roy. Soc. (London) **A215**, 291 (1952).

coupling to simulate Pb or Hg. In this calculation the energy gap was nearly as large as that of Pb or Hg, but the critical field was much closer to that of a weak coupling superconductor. The difference between these two calculations must be due to the different functional forms for $\Delta(\epsilon)$. To test this we report here the effect on the critical field of using several simple forms for $\Delta(\epsilon)$. The results are given in the first section.

In Sec. II, we give numerical solutions for $\Delta(\epsilon)$ using four different interactions for comparison with the requirements of the tunneling experiments and of the critical field.

I. THE CRITICAL FIELD

The most dramatic way of pointing out the peculiarity of the temperature dependence of the critical field of Pb or Hg in comparison to the weak coupling superconductors is by plotting the deviation from a parabolic form. The deviations are small, being less than 5 or 6%; however, Pb and Hg have positive deviations while the weak coupling superconductors have negative deviations.

In Finnemore and Mapother's[9] calculation of the critical field deviation, which they compared with their Hg results, they made the following three assumptions: First, that $\Delta(\epsilon,T)$ is a constant independent of ϵ for each temperature. Second, that the temperature dependence of $\Delta(T)/\Delta(0)$ is the same as for the weak coupling limit of BCS. Third, that the energy gap to critical temperature, $2\Delta(0)/kT_c$, is an adjustable parameter. We discuss these assumptions below in the light of the solutions for various interactions. They then found that they obtained roughly the experimental maximum deviation[9,11] of $+0.015$ and at roughly the experimental reduced temperature of 0.5 if they chose the energy gap to be 4.1 kT_c. This is in fair agreement with the value of $4.6kT_c$ found by infrared absorption[12] and $4.1kT_c$ found by tunneling.[13]

The result above is in sharp contrast to the critical field calculation of the author[4] in which an actual solution of the BCS integral equation was used for the energy-gap function. The interaction used was a square-well in $|\epsilon-\epsilon'|$ and was modeled after the Bardeen-Pines[10] type of interaction. The strength of the interaction was adjusted to give a gap of $4.0kT_c$ and the resulting critical field had a maximum deviation from a parabolic form of -0.026 and this occurred at a reduced temperature of about 0.7. Thus, the critical field was very similar to the weak coupling result of BCS even though the energy gap to kT_c was almost as large as that used by FM.

There are two possible reasons for the disagreement between these calculations. First, in the expression used by FM the integration is taken to infinite energy with the energy-gap function a nonvanishing constant over the entire range, whereas, in our calculation, the energy-gap function rapidly approached zero a few times the Debye energy from the Fermi surface.[14] We are not dealing with the weak coupling limit, since with Hg the Debye energy is roughly only 7.5 times $\Delta(0)$; hence, it would seem appropriate to repeat the calculation of FM, but with the integration cut off at about eight times $\Delta(0)$.

The second difference between the two energy-gap functions is in their behavior near the Fermi surface—that of FM is constant while ours is of the form

$$\Delta(\epsilon)=\Delta(0)[1-C\epsilon^2], \tag{1}$$

with $C>0$ and of the order of $(1/k\theta_D)^2$.

To see the effect of a variation in the functional form of $\Delta(\epsilon)$ in the energy region near or larger than $k\theta_D$, we have computed the critical field using a constant $\Delta(\epsilon)$ out to an energy ϵ_m and then zero beyond. Otherwise, we have retained the same three assumptions of FM. We have considered three cases with $\epsilon_m=16\Delta_0$, $8\Delta_0$, and $4\Delta_0$, all with $2\Delta_0/kT_c=4.1$. The critical field was obtained by numerically integrating the expressions in Eqs. (24), (25), and (26) of reference 4. The constant solid curves that are cut off at 4 and at 8 in Fig. 1 represent two of the three functional forms for $\Delta(\epsilon)$. There was very little difference between the deviation functions of these three examples, the maximum deviations being $+0.0203$, $+0.0198$, and $+0.0185$ for $\epsilon_m=16$, 8, and $4\Delta_0$, respectively, with this maximum occurring at very nearly the same reduced temperature for all three. The deviation function for the one case with $\epsilon_m=8\Delta_0$ is plotted in Fig. 2 and labeled "constant Δ."

We conclude that the critical field curve for Pb or Hg is rather insensitive to the functional form of $\Delta(\epsilon)$ for energies in the region of or larger than the experimental value of $k\theta_D$. It should be stressed that the constant Δ used here and by FM is not a solution of the BCS integral equation using a constant interaction, V. With a constant V it is impossible to have a solution with $2\Delta_0/kT_c$ larger than 4.0, and the value of four occurs only in the strong coupling limit. For the constant V of a value such that $2\Delta_0/kT_c$ equals 3.9, ϵ_m is only $0.7657\Delta_0$[15] and the critical field deviation function is negative with the largest deviation being -0.063. Such a critical field curve is quite different from that found by FM for the case $2\Delta_0=3.9kT_c$. The reason for the large difference is that $\Delta(\epsilon)$ has been changed near $\epsilon=0$.

To check further the effect of changes in $\Delta(\epsilon)$ near the Fermi surface, we have considered energy-gap functions of the form of Eq. (1) for various values of the

[11] J. E. Schirber and C. A. Swenson, Phys. Rev. **123**, 1115 (1961).
[12] P. L. Richards and M. Tinkham, Phys. Rev. **119**, 575 (1960).
[13] J. M. Rowell and A. G. Chynoweth, Bull. Am. Phys. Soc. **7**, 473 (1962).

[14] The energy-gap functions we consider in this paper are symmetrical with respect to the Fermi surface; so we need consider only energies larger than the Fermi energy. When we speak of an energy ϵ smaller than the Debye energy, we mean $|\epsilon|<k\theta_D$, where ϵ is with respect to the Fermi surface.
[15] P. M. Marcus (private communication).

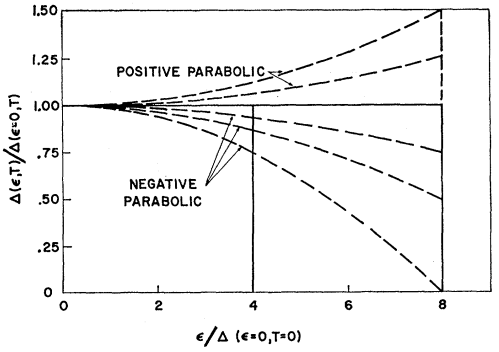

FIG. 1. Energy dependence of simple energy-gap functions used in the critical field calculations of Fig. 2.

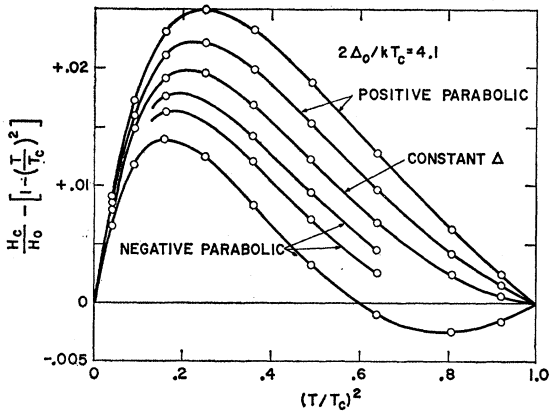

FIG. 2. Calculated temperature dependence of the critical field deviation function using the energy-gap functions of Fig. 1.

constant C. For all our cases, we have taken $\Delta(\epsilon)\equiv 0$ for $\epsilon>8\Delta_0$. The different examples considered are plotted with the dashed lines in Fig. 1. The cases with $C>0$, we denote as "negative parabolic" (since the quadratic term is negative), while those with $C<0$, we denote as "positive parabolic." Although Eq. (1) is a very simple functional form, it should represent any analytic $\Delta(\epsilon)$ near the Fermi surface since it is the first two terms of an expansion in powers of ϵ. As we mentioned above, there is no linear term in the expansion. Equation (1) would not be a good representation of an arbitrary $\Delta(\epsilon)$ far from the Fermi surface, but we have already found that values at large ϵ do not have much effect on the critical field.

In order to calculate the critical field, we take over the second and third assumptions of FM. We change the first to assuming that $\Delta(\epsilon,T)/\Delta(\epsilon=0,T)$ is independent of temperature. That is, the form of Δ as a function of ϵ is independent of T. This was found to be very nearly true for actual solutions of the integral equation.[4,5]

Figure 2 gives the deviation function for the different parabolic forms for the case $2\Delta_0=4.1kT_c$. The negative parabolic $\Delta(\epsilon)$ of Fig. 1 that deviates the least from the constant Δ has very nearly the same critical field as the constant Δ cut off at $4\Delta_0$. Thus, adding this small amount of a parabolic factor has the same effect as eliminating the nonvanishing $\Delta(\epsilon)$ from 4 to 8. The trend of the deviation function on adding more of a negative parabolic part to $\Delta(\epsilon)$ is in the right direction to explain the different critical field curves of FM and of the author.[4] In the latter calculation, $\Delta(\epsilon)$ had a large negative parabolic part that would take $\Delta(\epsilon)$ to zero at about $2.4\Delta_0$.

If the three assumptions used here are correct, then the positive critical field deviation functions of Pb and Hg indicate that $\Delta(\epsilon)$ has very little negative parabolic part. That is, $\Delta(\epsilon)$ is either very nearly a constant or has a positive parabolic part near the Fermi surface. A similar conclusion had previously been arrived at by the author[4] on considering the jump in the specific heat in Pb at T_c. However, at that time, it was incorrectly conjectured that solutions of the integral equation could not have a positive parabolic part. Subsequently, it was shown[5,6] that the Eliashberg type of interaction does give such a solution.

II. SOLUTIONS OF THE INTEGRAL EQUATION

Now we consider actual spherically symmetrical solutions of the BCS integral equation

$$\Delta(\epsilon)=-\tfrac{1}{2}\int d\epsilon' V(\epsilon,\epsilon')\frac{\Delta'}{E'}\tanh(\tfrac{1}{2}\beta E'), \qquad (2)$$

for the various effective electron-electron interactions $V(\epsilon,\epsilon')$. Here V is the usual $V_{\mathbf{k}\mathbf{k}'}$ averaged over angles and multiplied by the density of states (in the normal state), while

$$E'\equiv[\epsilon'^2+\Delta(\epsilon')^2]^{1/2} \qquad (2a)$$

is the quasiparticle excitation energy. We have carried out the calculation on an IBM 7090 by a "quasilinearization" process described earlier.[4]

In Bogoliubov's derivation[3] of the superconductivity theory by cancellation of dangerous graphs, he arrives at an integral equation of the form (2) with $V_{\mathbf{k}\mathbf{k}'}$ given by

$$V_{\mathbf{k}\mathbf{k}'}=-\frac{|M_{\mathbf{q}}|^2}{\hbar\omega_{\mathbf{q}}+E_{\mathbf{k}}+E_{\mathbf{k}'}}, \qquad (3)$$

where $M_{\mathbf{q}}$ is the matrix element for the scattering of an electron from $\mathbf{k}$ to $\mathbf{k}'$ with emission or absorption of a phonon of wave vector $\mathbf{q}$ and energy $\hbar\omega_{\mathbf{q}}$.[16]

[16] In the Bogoliubov theory the excitation energies are given by Eq. (2a) only in the weak coupling limit and then only for excitations near the Fermi surface. The more complete expression for the energy [see Bogoliubov, Tolmachev, and Shirkov, reference 3. Eq. (3.8)] involves integrals with singularities near the Debye energy. These singularities will have an effect on the density of states and possibly also on the critical field. Our calculation is of a simplified version of the Bogoliubov theory which ignores these singularities. The same simplification was made by Bogoliubov in reference 3. A better treatment of the Bogoliubov theory at the coupling strengths corresponding to Pb and Hg should also take into account higher order terms in the perturbation expansion as pointed out by G. Rickayzen, Phys. Rev. **111**, 817 (1958).

For our calculation we have assumed an Einstein phonon spectrum so that ω_q is independent of $\mathbf{q}$. Integrating over angles and adding a positive constant term for the screened Coulomb interaction, we have

$$V(\epsilon,\epsilon')=-\frac{A}{1+E+E'}+B,$$

for the Bogoliubov interaction in energy units of $\hbar\omega=k\theta_D$. Here A and B are constants representing the strength of the two interactions and we have treated them as parameters. We have cut off the interaction at ϵ' equal to about four times $k\theta_D$. In Fig. 3 is plotted the solution $\Delta(\epsilon)$ at $T=0$ for this interaction with $A=0.6$ and $B=0.3$; because of the symmetry, we need integrate Eq. (2) only over positive ϵ' if we drop the $\frac{1}{2}$ factor. For this example $T_c/\theta_D=0.0074$, while $\Delta(\epsilon,T)/\Delta(0,T)$ for any particular ϵ changed by less than 2% over the entire temperature range. Table I gives the results for three coupling strengths.

For comparison, we give in Table I and Fig. 3 results previously reported[4] for the Bardeen-Pines interaction[10] with a Debye phonon spectrum. The interaction used for this case is not the square well in $|\epsilon-\epsilon'|$, but is that given in Eq. (12) of reference 4. The singularity in the potential was arbitrarily cut off for convenience since lifetime effects would cut it off anyway.

Eliashberg[17] has obtained an effective electron-electron interaction starting with the assumption that the electrons are in a BCS state. Using this interaction, we have obtained numerical solutions of the integral equation (2) for both the Einstein phonon spectrum and the Debye spectrum. For the former, the interaction is (the integration over angles is trivial)

$$V(\epsilon,\epsilon')=-\tfrac{1}{2}A\left\{\frac{1}{1+E+E'}+\frac{1}{1-E+E'}\right\}+B. \quad (4)$$

With the Debye spectrum the integration over angles gives

$$V(\epsilon,\epsilon')=-A\{F(E'+E)+F(E'-E)\}+B, \quad (5)$$

with

$$F(x)=\frac{a_1-a_2x+x^2\ln|1+1/x|}{a^2+x^2}, \quad (6)$$

where

$$a_1=\tfrac{1}{2}a^2\ln(1+1/a^2),$$
$$a_2=a\cot^{-1}a,$$

while a^2 is proportional to the interelectron spacing and is of the order of 0.4 for most superconductors. We used $a^2=0.4$ for our calculations. In Eqs. (4) and (5), the constants A and B are the strengths of the electron-phonon interaction and the electron-electron Coulomb interaction, respectively. In obtaining (5), we have used the "jellium" model for the electron-phonon interaction.

Morel and Anderson[6] found approximate analytic solutions at $T=0$ for the interaction (4), while Culler *et al.*[5] have used an interaction that is very similar to (5) to find numerical solutions over the temperature range up to T_c. Our results, which are shown near the Fermi surface in Fig. 3 and over a larger energy range in Figs. 4 and 5, are nearly identical to these previous calculations. The solutions for the two different phonon spectra are also quite similar to each other.

In these calculations we cut off the interaction for ϵ' larger than ϵ_m where we used $\epsilon_m=18.8k\theta_D$ for the cases with the Einstein spectrum and $\epsilon_m=10k\theta_D$ for the

TABLE I. The ratio of the critical temperature to Debye temperature and of the energy gap to critical temperature for different coupling strengths and different interactions. The ratio T_c/θ_D increases with increasing coupling.

T_c/θ_D	$2\Delta_0/kT_c$	$\bar{\Delta}(k\theta_D)/\Delta_0$
Bogoliubov interaction, Einstein phonon spectrum		
$5.76\ 10^{-5}$	3.52	−66.5
0.00740	3.39	−31.4
0.167	2.70	− 1.22
Bardeen-Pines interaction, Debye phonon spectrum		
0.00720	3.48	− 9.17
Eliashberg interaction, Einstein phonon spectrum		
0.00068	3.52	+ 3.13
0.0148	3.43	+ 2.48
0.0366	3.33	+ 2.25
Eliashberg interaction, Debye phonon spectrum		
0.000855	3.51	+16.9
0.00898	3.40	+11.0
0.07	...	+ 5.8

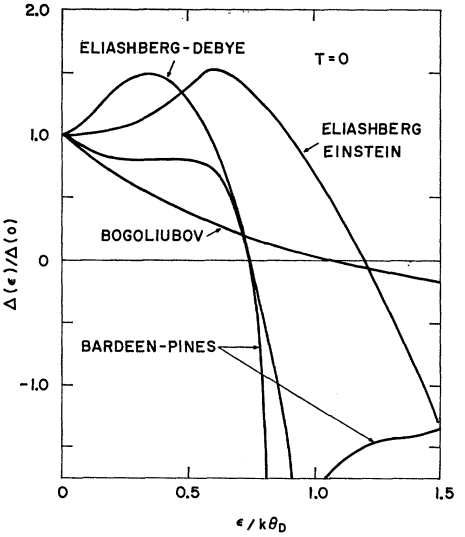

FIG. 3. Numerical solutions of the BCS integral equation for various interactions and at $T=0$. These curves show the energy dependence near the Fermi surface. The functions $\Delta(\epsilon,T)/\Delta(0,T)$ were found to be nearly identical to these.

[17] G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **38**, 966 (1960) [translation: Soviet Phys.—JETP **11**, 696 (1960)].

Debye cases. The qualitative features of the solutions do not seem to be dependent on the precise value of ϵ_m. The singularities in the interactions have also been cut off. For the Debye case the logarithm in the interaction was treated as constant in an energy region of width $0.05k\theta_D$ about the singularity. With the Einstein spectrum the singularity is of the $1/x$ type which is stronger than the logarithmic singularity in the other case. Because of this, we cut off the singularity over a wider region, $0.5k\theta_D$, and joined the positive value of the interaction on one side with the negative value on the other side by a linear interpolation. For a narrower cut-off region of the singularity, we found the resonances at multiple values of $k\theta_D$ as discussed by Rowell, Chynoweth, and Phillips,[18] but we have not made accurate calculations of these effects. In order to make a quantitative comparison with the experiments, it would be necessary to know in detail how the lifetime effects cut off the singularity and this has not been worked out as yet.

To take into account the fact that the Coulomb interaction averaged over angles is not a constant but decreases for large $\mathbf{k}$ or $\mathbf{k}'$, we have replaced the constant B in the interactions (4) and (5) by

$$V^{\mathrm{Coul}}(\epsilon,\epsilon')=B/\{[1+|\epsilon|/\epsilon_F]^{1/2}[1+|\epsilon'|/\epsilon_F]^{1/2}\}, \quad (7)$$

where ϵ_F is a constant. If we take for ϵ_F the Fermi energy $\hbar^2k_F^2/2m$, then for ϵ and ϵ' above the Fermi level, Eq. (7) is of the form

$$V^{\mathrm{Coul}}=Bk_F^2/(kk').$$

We used $\epsilon_F=100k\theta_D$ in our calculations. The effect of this variation of V^{Coul} on the solution was to cause $\Delta(\epsilon)$ for ϵ larger than $4k\theta_D$ to be slowly increasing toward zero instead of having a constant negative value (see Figs. 4 and 5).

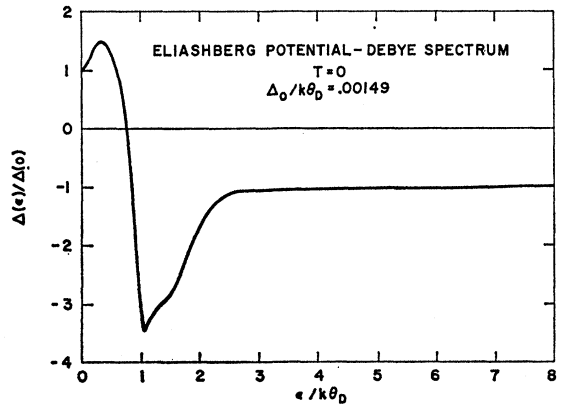

FIG. 4. Numerical solution of the BCS integral equation for the Eliashberg interaction with a Debye phonon spectrum. This is the same solution as in Fig. 3, but it is shown over a la ger energy range.

[18] J. M. Rowell, A. G. Chynoweth, and J. C. Phillips, Phys. Rev. Letters **9**, 59 (1962).

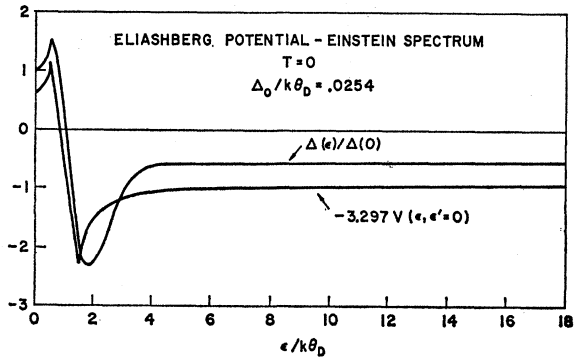

FIG. 5. Numerical solution of the BCS integral equation for the Eliashberg interaction with an Einstein phonon spectrum. This is the same solution as in Fig. 3. Also plotted is $C_1V(\epsilon,\epsilon'=0)$, where C_1 and C_2 are chosen to satisfy Eq. (8) at the energy ends.

One of the striking features of the calculations of Culler *et al.*[5] is that for their plot of $\Delta(\epsilon)$ with the Coulomb repulsion included (their Fig. 4 with C not zero) there is a second relative maximum at $\epsilon\sim3k\theta_D$ and the solution even becomes positive at this point. For our calculation in Fig. 4, there is also a relative maximum at $\epsilon=3.5k\theta_D$, but it is so slight that it does not show up on the graph. For our strongest coupling case in Table I, there is a pronounced maximum at $3.3k\theta_D$, but the function does not become positive. When the singular peak in the interaction was smeared over a region larger than $0.05k\theta_D$,this bump in $\Delta(\epsilon)$ became smaller. Thus, this second maximum is another example of the resonances of Rowell *et al.*[18]

Morel and Anderson[6] found their solutions of the BCS integral equation by assuming a solution of the form

$$\Delta(\epsilon)/\Delta_0=C_1V(\epsilon,\epsilon'=0)+C_2, \quad (8)$$

and determining the constants C_1, C_2, and Δ_0 so that $\Delta(0)=\Delta_0$ and also so that the equation is satisfied at high and low energies. In Fig. 5 we have plotted together with our solution $\Delta(\epsilon)/\Delta_0$ the function $C_1V(\epsilon,\epsilon'=0)$ where C_1 has been chosen $C_1=-3.297$. With this value of C_1, Eq. (8) is satisfied at the two ends of the energy range with $C_2=+0.3805$. If Eq. (8) were satisfied, the one curve of Fig. 5 would be a constant amount below the other over the entire range. The deviations from (8) occur in the region of the singularity; this is just the region where Morel and Anderson indicated their solution would be inaccurate. We can not compare our values of Δ_0, C_1, and C_2 with those using the method of Morel and Anderson since we have treated the singularity quite differently. Figure 5 demonstrates the way in which we have cut off this singularity in V.

Equation (8) not only gives the qualitative features of $\Delta(\epsilon)/\Delta_0$ for the Eliashberg interaction with the Einstein spectrum but also for this interaction with the Debye spectrum. With the Bogoliubov interaction, the agreement of the numerical solution with Eq. (8) is very

good over the entire energy range. However, with the Bardeen-Pines interaction, or even a square well in $|\epsilon-\epsilon'|$, the solution does not qualitatively agree with (8) in the neighborhood of the Fermi surface.

We are now in a position to compare the results of our numerical calculations with the experimental requirements. First of all, all of the interactions give the BCS results in the weak coupling limit and, thus, agree with the experimental law of corresponding states. When the screened Coulomb interaction is included, all the solutions change with energy from a positive solution with a negative slope to a negative solution with negative slope in the region of $k\theta_D$. This is the requirement from the tunneling results of Giaever *et al.*[8]; however, the slope for the simplified Bogoliubov interaction is so small that it does not appear that it could give the magnitude of Giaever's effect.[16] The resonances of Rowell *et al.*[18] occur only for the interactions with singularities; this also rules against the simplified Bogoliubov interaction.

In comparing with the critical field results of FM, we find that as long as the coupling is not too large the first two assumptions we made about the solutions are valid. That is, $\Delta(\epsilon,T)/\Delta(\epsilon=0, T)$ is essentially independent of temperature, and $\Delta(\epsilon=0, T)/\Delta(\epsilon=0, T=0)$ is very nearly the same as the BCS weak coupling result. The latter function is very close to the BCS strong coupling result

$$\Delta(T)/\Delta(0)=\tanh[(T_c/T)(\Delta(T)/\Delta(0))], \qquad (9)$$

as has been observed before.[4,5]

For comparing the calculated $\Delta(\epsilon)$ with the requirement we found from the critical field data, we have given in the third column of Table I the value of $\bar{\Delta}(k\theta_D)/\Delta_0$ where the function $\bar{\Delta}(\epsilon)$ is defined as the quadratic function (1) which fits the actual numerical solution near the Fermi surface. The values in Table I should be compared with the values of $\Delta(\epsilon)/\Delta_0$ in Fig. 1 for $\epsilon=8\Delta_0$, since for our critical field calculations of the previous section we assumed $k\theta_D\sim 8\Delta_0$.

We see from Table I that the Bogoliubov and Bardeen-Pines interactions produce large negative parabolic parts in $\Delta(\epsilon)$ near the Fermi surface. This negative part is the smallest in the stronger coupling Bogoliubov interaction, but even here it appears to be too much to account for the positive critical field deviation function of Pb and Hg.[16] On the other hand, the Eliashberg interaction gives a positive parabolic part to $\Delta(\epsilon)$ as shown in earlier calculations,[5,6] and for the stronger coupling cases the magnitude of this parabolic part is not unreasonable. The conclusion was previously reached,[4] on the basis of the magnitude of the jump in the specific heat of Pb at T_c, that $\Delta(\epsilon)$ must have a positive parabolic part. These two pieces of information both favor the Eliashberg interaction over the other two.

There is still a problem about the third assumption in our calculation of the critical fields, namely, that $2\Delta_0/kT_c$ takes values as large as 4.1. The solution of the integral equation sets the value of this ratio; so it is not an adjustable parameter. The weak coupling solution gives 3.5 for the ratio for every interaction tried. Whereas with the constant interaction of BCS and also with a square well in $|\epsilon-\epsilon'|$ the ratio increased with increasing coupling,[4] we see from Table I that the opposite is true for the more realistic interactions. This was previously noted by Culler *et al.*[5] Thus, we can not obtain a positive deviation in the critical field with the solutions from the Eliashberg interaction because $2\Delta_0/kT_c$ is not large enough. It may be that the large value of the ratio is brought about by the things that were neglected in the present calculation such as lifetime effects, anisotropies, or other crystal effects. If these other effects do produce a large energy gap to critical temperature ratio without changing the validity of the first two assumptions, then the critical field data favor the Eliashberg interaction.

The Bardeen-Pines interaction was derived[10] on the assumption that the electrons are in the normal state. Hence, it is not surprising that this interaction is not as accurate as the other two for the superconducting state. Comparing the Eliashberg and Bogoliubov interactions, we have seen that the tunneling experiments as well as the critical field and specific heat data point to the Eliashberg interaction as being the better. On the other hand, Liu[19] found that the Bogoliubov interaction led to a lower energy state than the Eliashberg interaction and thus should be a better model for the superconducting state. This discrepancy is as yet unresolved.

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Note added in proof. Since this paper was submitted for publication, J. R. Schrieffer [D. J. Scalapino, and J. W. Wilkins, Phys. Rev. Letters **10**, 336 (1963)], have shown that it is not the ordinary quasiparticle density of states in the superconductor that is appropriate for the tunneling current. They have also shown that the imaginary part of the energy-gap function is as important in explaining the tunneling anomaly as is the real part.

[19] S. H. Liu, Phys. Rev. **125**, 1244 (1962).